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LETTER TO THE EDITOR

Lie and Noether symmetries and a result of Logan

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Abstract. The notions of Lie and conformal invariance symmetry for a regular second-order equation field are shown to be essentially identical. It is pointed out that the way in which conformal invariance symmetries are traditionally defined is not invariant and that this leads to difficulties in interpreting a recent result of Logan. However, the introduction of coordinate-free geometrical apparatus leads to a simple proof of Logan's result and interpretation of the examples considered by him.

Recently, there appeared in this journal a theorem and its converse which attempted to clarify the distinction between two kinds of symmetry which can be defined in a regular particle-type variational problem (Logan 1985). Unfortunately one of these kinds of symmetries, known as a conformal invariance symmetry, is not well defined. It depends on the choice of a function or functions, the vanishing of which determines the differential equation(s). I shall argue in this letter that there is a well defined invariant concept of conformal invariance symmetry, but whose definition appears to take a rather different form. Given this definition and some other fairly standard facts from Lagrangian theory, Logan's result amounts to the fact that every Noether symmetry of a variational problem is a Lie symmetry. The converse, however, does not hold and Logan's result attempts to throw some light on the question of when a Lie symmetry is Noether. I shall give two necessary and sufficient conditions for this and use them to explain the two examples considered by Logan.

Another related issue is the perhaps somewhat paradoxical fact that when Euler-Lagrange equations are considered from the point of view of the theory of jet bundles (Sniatycki 1970, Crampin *et al* 1984), they are often formulated as a vector field or differential system on the *first*-order jet bundle, though the Euler-Lagrange equations are themselves *second* order.

I shall use modern geometric apparatus which I hope complements the classical analytical methods and serves to focus some of the issues more sharply. The notation agrees with that used in Crampin *et al* (1984). In particular, I denote the interior product of a p-form α by a vector field X by $X \lrcorner \alpha$ and the Lie derivative of α along X by $L_X\alpha$.

In Crampin *et al* (1984), second-order equation fields were discussed in terms of jet bundles. It is worthwhile to review and extend the remarks here. Suppose that an m-manifold M is the configuration space of some Lagrangian system. Then it is well known that time-dependent Lagrangian theory leads to a vector field—the Euler-Lagrange field on $\mathbb{R} \times TM$ which is the evolution space of the system. Now $\mathbb{R} \times TM$ can be identified with $J^1(\mathbb{R}, M)$, the bundle of 1-jets of local smooth curves on M. If coordinates (t, x^i) are chosen for $\mathbb{R}\times M$, there are naturally induced coordinates (t, x^i, u^i) for $J^1(\mathbb{R}, M)$. Given the Lagrangian $L: J^1(\mathbb{R}, M)\to\mathbb{R}$, the Euler–Lagrange equations associated with L may be considered in two different, but entirely equivalent ways; firstly, as the second-order equation field

$$\Gamma = \frac{\partial}{\partial t} + u^i\frac{\partial}{\partial x^i} + \Gamma^i\frac{\partial}{\partial u^i}$$

(where the Γ^i are the Euler–Lagrange expressions) i.e. explicitly

$$\Gamma^i = \left(\frac{\partial^2 L}{\partial u^i\,\partial u^j}\right)^{-1}\left(\frac{\partial L}{\partial x^j} - \frac{\partial L}{\partial t\,\partial u^j} - u^k\frac{\partial^2 L}{\partial u^j\,\partial x^k}\right)$$

and secondly, as the module of 1-forms generated by $\mathrm{d}x^i - u^i\,\mathrm{d}t$ and $\mathrm{d}u^i - \Gamma^i\,\mathrm{d}t$. From the first point of view, a solution to the Euler–Lagrange equations is simply an integral curve of Γ and from the second, an integral manifold of a $2m$-dimensional Pfaffian module subject also to the transversality condition $\mathrm{d}t \neq 0$.

The preceding discussion is of course not limited to Euler–Lagrange fields. If the quantities Γ^i were simply functions of t, x^i and u^i then Γ would correspond to an arbitrary second-order equation field. One might well ask why second-order equation fields give rise to vector fields and Pfaffian modules on the *first*-order jet bundle $J^1(\mathbb{R}, M)$. The reason is the following. A (determined) system of second-order ordinary differential equations actually determines, at least locally, a codimension m submanifold Σ of $J^2(\mathbb{R}, M)$, the *second*-order jet bundle of local smooth curves on M. Given the coordinates (t, x^i) on $\mathbb{R}\times M$, there are naturally induced coordinates (t, x^i, u^i, v^i) on $J^2(\mathbb{R}, M)$. Σ is determined locally by the vanishing of the m quantities $v^i - \Gamma^i$. Actually, I am assuming here that the second-order system is *regular*, that is to say, that Σ is transverse to the fibration $J^2(\mathbb{R}, M)\to J^1(\mathbb{R}, M)$. If this is so, then by the implicit function theorem, the equations determining Σ may be solved for the fibre coordinates of $J^2(\mathbb{R}, M)$ over $J^1(\mathbb{R}, M)$, namely the v^i. (I use the term 'regular' advisedly here, because in the case where the second system is actually a Euler–Lagrange system, regularity is equivalent to the invertibility of the matrices $\partial^2 L/\partial u^i\,\partial u^j$, which is just the usual notion of regularity for a Euler–Lagrange system.) A regular second-order system is, then, precisely a section of $J^2(\mathbb{R}, M)$ over $J^1(\mathbb{R}, M)$. Thus Σ may be identified via this section with $J^1(\mathbb{R}, M)$. Moreover, the Pfaffian system $\{\mathrm{d}x^i - u^i\,\mathrm{d}t, \mathrm{d}u^i - v^i\,\mathrm{d}t\}$ restricted to Σ, integral curves of which determine solutions of the second-order system, descends to the Pfaffian system $\{\mathrm{d}x^i - u^i\,\mathrm{d}t, \mathrm{d}u^i - \Gamma^i\,\mathrm{d}t\}$ on $J^1(\mathbb{R}, M)$.

I shall now show how the preceding remarks serve to clarify the distinction between two kinds of symmetry which may be defined for regular second-order equation fields. First of all, however, it is necessary to recall the prolongation construction for vector fields. If $X = \tau\,\partial/\partial t + \xi^i\,\partial/\partial x^i$ is a vector field on $\mathbb{R}\times M$, then there is a unique vector field $X^{(r)}$ on $J^r(\mathbb{R}, M)$ (the rth-order jet bundle), characterised by the property that it projects onto X and preserves the rth-order contact module $\Omega^{(r)}$, in the sense that $L_{X^{(r)}}\Omega^{(r)} \subset \Omega^{(r)}$. For $r = 1$ and 2 respectively, the cases of interest here, one has

$$X^{(1)} = \tau\frac{\partial}{\partial t} + \xi^i\frac{\partial}{\partial x^i} + (\dot\xi^i - \dot\tau u^i)\frac{\partial}{\partial u^i}$$

$$X^{(2)} = \tau\frac{\partial}{\partial t} + \xi^i\frac{\partial}{\partial x^i} + (\dot\xi^i - \dot\tau u^i)\frac{\partial}{\partial u^i} + (\ddot\xi^i - 2\dot\tau v^i - \ddot\tau u^i)\frac{\partial}{\partial v^i}$$

where

$$\dot{\xi}^i = \frac{\partial \xi^i}{\partial t} + u^k \frac{\partial \xi^i}{\partial x^k} \qquad \ddot{\xi}^i = \frac{\partial \dot{\xi}^i}{\partial t} + u^k \frac{\partial \dot{\xi}^i}{\partial x^k} + v^k \frac{\partial \dot{\xi}^i}{\partial u^k}$$

and likewise for $\dot{\tau}$ and $\ddot{\tau}$. Of the two notions of symmetry which I referred to above, the first is a Lie symmetry which is a vector field X on $\mathbb{R} \times M$ such that $[\Gamma, X^{(1)}] = \lambda \Gamma$ (for some function λ on $J^1(\mathbb{R}, M)$ which may actually be shown to be $\dot{\tau}$). Equivalently, $X^{(1)}$ stabilises the exterior differential system $\mathscr{D}$ generated by the Pfaffian forms $\{\mathrm{d}x^i - u^i\,\mathrm{d}t, \mathrm{d}u^i - \Gamma^i\,\mathrm{d}t\}$; that is to say $L_{X^{(1)}}\mathscr{D} \subset \mathscr{D}$ where $\mathscr{D}$ is the algebraic ideal generated by $\{\mathrm{d}x^i - u^i\,\mathrm{d}t, \mathrm{d}u^i - \Gamma^i\,\mathrm{d}t\}$ and their exterior derivatives. This latter definition is both easy to compute with and generalises easily to systems of partial differential equations.

The second kind of symmetry I wish to consider is what I shall call a conformal invariance symmetry. Suppose that the submanifold Σ of $J^2(\mathbb{R}, M)$ is determined by the vanishing of the m functions $F^1, \ldots, F^m$. Then a vector field X on $\mathbb{R} \times M$ is said to be a conformal invariance symmetry if the zero locus of $X^{(2)}F^1, \ldots, X^{(2)}F^m$ coincides with Σ. More geometrically and invariantly, a conformal invariance symmetry is precisely a vector field of the form $X^{(2)}$ which is tangential to Σ. The flow of $X^{(2)}$ therefore determines a 1-parameter group of diffeomorphisms of Σ, or, less precisely, the flow of $X^{(2)}$ leaves Σ invariant.

The definition of conformal invariance symmetry just given should be compared to the manifestly non-invariant definitions of, for example, Ames (1972), Bluman and Cole (1974) and Logan (1985). These authors define a conformal invariance symmetry X as one which satisfies $X^{(2)}F^i = \alpha^i_j F^j$, where Σ is determined by the conditions $F^i = 0$ and the α^i_j are real-valued functions on $J^2(\mathbb{R}, M)$. To see the difficulties caused by this definition consider, for example, the equation $\ddot{x} = 0$ as a codimension one submanifold of $J^2(\mathbb{R}, \mathbb{R})$. Then using coordinates (t, x, u, v), Σ is determined by the condition $v = 0$ and it is easy to see that $X = t\,\partial/\partial t + u\,\partial/\partial u$ is a conformal invariance symmetry according to the definition given here. Indeed, this is apparent from the fact that $X^{(2)} = t\,\partial/\partial t + u\,\partial/\partial u - 2v\,\partial/\partial v$. However, Σ is equally well specified by the condition $v^3 + v = 0$ and according to the alternative definition, X would not qualify as a conformal invariance symmetry.

Proposition 1. X is a Lie symmetry of a regular second-order equation iff it is a conformal invariance symmetry.

Proof. Suppose that the codimension m submanifold Σ of $J^2(\mathbb{R}, M)$ is determined by the vanishing of the m functions $F^1, \ldots, F^m$ where $F^i = v^i - \Gamma^i$, (v^i) are coordinates of the fibres of $J^2(\mathbb{R}, M)$ over $J^1(\mathbb{R}, M)$ and the Γ^i are functions of t, x^i and u^i, (t, x^i, u^i) being a coordinate system on $J^1(\mathbb{R}, M)$. (This description of Σ is possible because Σ is assumed to be regular.) Let $X = \tau\,\partial/\partial t + \xi^i\,\partial/\partial x^i$ be a vector field on $\mathbb{R} \times M$. Then by direct calculation one finds

$$X^{(2)}F^i = \ddot{\xi}^i - 2\dot{\tau}v^i - \ddot{\tau}u^i - X^{(1)}\Gamma^i \tag{1}$$

where $X^{(1)}$ on $J^1(\mathbb{R}, M)$ is abusively identified with a vector field on $J^2(\mathbb{R}, M)$ (this abuse is permissible because the Γ^i are functions on $J^1(\mathbb{R}, M)$).

Now one finds that X is a Lie symmetry iff

$$X^{(1)}\Gamma^i = \ddot{\xi}^i - 2\dot{\tau}\Gamma^i - \ddot{\tau}u^i. \tag{2}$$

(One computes $[\Gamma, X]$ and sees that $[\Gamma, X]=\lambda\Gamma$ iff $\lambda=\dot{\tau}$ and (2) is satisfied.) Thus, if X is a Lie symmetry

$$X^{(2)}F^i=2\dot{\tau}(\Gamma^i-v^i)=-2\dot{\tau}F^i \tag{3}$$

and hence X is a conformal invariance symmetry. Conversely, if X is a conformal invariance symmetry, then on Σ, $X^{(2)}F^i=0$ and hence since $v^i=f^i$, (1) implies that X is a Lie symmetry.

The import of this proposition and the remarks preceding it is that a conformal invariance symmetry is much more naturally viewed as a symmetry of the second-order equation field Γ or, equivalently, as a symmetry of the associated exterior differential system determined by the annihilator of Γ. It is worthwhile to note that these considerations apply to arbitrary (regular) second-order equations and not just Euler–Lagrange equations. By contrast, I shall consider the next Noether symmetries which are only defined for Euler–Lagrange systems.

Let $L: J^1(\mathbb{R}, M)\to\mathbb{R}$ be a regular Lagrangian and denote the corresponding Euler–Lagrange field by Γ_L. Let Θ_L denote the Cartan 1-form associated with L. In fact, choosing coordinates (x^i) on M and denoting the natural coordinate on $\mathbb{R}$ by t and the induced coordinates on the fibres of $J^1(\mathbb{R}, M)$ over $\mathbb{R}\times M$ by (u^i), $\Theta_L=L\,dt+(\partial L/\partial u^i)(dx^i-u^i\,dt)$. Evidently there is a bi-unique correspondence between (not necessarily regular) Lagrangians and Cartan forms; indeed, this correspondence is even an isomorphism of real vector spaces. (For more on the approach to Euler–Lagrange equations using the Cartan form, see Crampin (1977), Sarlet and Cantrijn (1982), Prince (1983) and Crampin *et al* (1984).) Γ_L is related to Θ_L by the conditions

$$\langle\Gamma, \Theta_L\rangle=L \tag{4}$$

$$\Gamma\,\lrcorner\, d\Theta_L=0. \tag{5}$$

Γ also satisfies $\Gamma(t)=1$ and this and (5) characterise Γ uniquely.

A Noether symmetry may now be defined as a vector field X on $\mathbb{R}\times M$ such that for some function $f: J^1(\mathbb{R}, M)\to\mathbb{R}$

$$L_{X^{(1)}}L\,dt=df. \tag{6}$$

It is appropriate to refer to a vector field X satisfying (6) as a Noether symmetry, since it is precisely the kind of symmetry considered by Noether in her seminal paper (Noether 1918). Logan (1985) refers to the action $L\,dt$ (or actually the associated functional) as being 'divergent invariant'. From (6) it easily follows that f must be a function on $\mathbb{R}\times M$ and if $X=\tau\,\partial/\partial t+\xi^i\,\partial/\partial x^i$ where τ is a function of t only, that f is a function of t only. It is well known that (6) implies that

$$L_{X^{(1)}}\Theta_L\equiv df \mod \Omega^{(1)} \tag{7}$$

where $\Omega^{(1)}$ is the module of contact 1-forms on $J^1(\mathbb{R}, M)$ (see, for example, Crampin 1977). In view of (7), some authors, for example Prince (1983), have proposed to weaken the definition of Noether symmetry to the following condition

$$L_{X^{(1)}}L\,dt\equiv df \mod \Omega^{(1)}. \tag{8}$$

However, this apparent generalisation is illusory because it still remains true that f is a function on $\mathbb{R}\times M$ and hence one can deduce that (8) implies (6). Indeed, this follows from part (ii) of the next proposition.

Proposition 2.
(i) If $X = \tau\, \partial/\partial t + \xi^i\, \partial/\partial x^i$ is a vector field on $\mathbb{R} \times M$, $L_{X^{(1)}}\Theta_L = \Theta_{X^{(1)}L + L\dot{\tau}}$.

(ii) X is a Noether symmetry, i.e. $L_{X^{(1)}}L\, dt = df$ for some $f: \mathbb{R} \times M \to \mathbb{R}$, iff $X^{(1)}L + L\dot{\tau} = \dot{f}$, iff $L_{X^{(1)}}\Theta = df$.

(iii) If X is a Noether symmetry, the corresponding first integral is $f - L\tau - X^V L$ where X^V is the vertical lift of X.

Proof. (i) I quote from Crampin *et al* (1984) where it is shown that $\Theta_L = L\, dt + S \circ dL$ where S is the fundamental 1-1 tensor on $J^1(\mathbb{R}, M)$ whose local expression is $\partial/\partial u^i \otimes (dx^i - u^i\, dt)$. Thus

$$
\begin{aligned}
L_{X^{(1)}}\Theta_L &= L_{X^{(1)}}(L\, dt + dL \circ S) \\
&= X^{(1)}L\, dt + L\, d\tau + d(X^{(1)}L \circ S) + dL \circ L_{X^{(1)}}S \\
&= X^{(1)}L\, dt + L\, d\tau + d(X^{(1)}L) \circ S + \dot{\tau}(dL \circ S) \qquad (\text{using } L_{X^{(1)}}S = \dot{\tau}S) \\
&= \Theta_{X^{(1)}L} + \Theta_{\dot{\tau}L} \\
&= \Theta_{X^{(1)}L + \dot{\tau}L}.
\end{aligned}
$$

(ii) Since $L_{X^{(1)}}L\, dt = X^{(1)}L\, dt + L\, d\tau$, it is clear that if X is a Noether symmetry, $X^{(1)}L + L\dot{\tau} = \dot{f}$. Next, note that by (i) $X^{(1)}L + L\dot{\tau} = \dot{f}$ iff $L_{X^{(1)}}\Theta_L = \Theta_{\dot{f}}$ which is easily seen to be equivalent to $L_{X^{(1)}}\Theta_L = df$. Finally, suppose that $L_{X^{(1)}}\Theta_L = df$. I shall show that X is a Noether symmetry. Now

$$
\begin{aligned}
L_{X^{(1)}}S \circ dL &= \dot{\tau}S \circ dL + S \circ d(X^{(1)}L) \\
&= \dot{\tau}S \circ dL + S \circ d(\dot{f} - L\dot{\tau}) \\
&= \dot{\tau}S \circ dL - \dot{\tau}S \circ dL + S \circ d\dot{f} - LS \circ d\dot{\tau} \\
&= 0
\end{aligned}
$$

(since S annihilates 1-forms semibasic with respect to the fibration $J^1(\mathbb{R}, M) \to \mathbb{R} \times M$). $L_{X^{(1)}}L\, dt = 0$ is now immediate from the fact that $\Theta_L = L\, dt + S \circ dL$.

(iii) The vertical lift X^V of a vector field X on $\mathbb{R} \times M$ to $J^1(\mathbb{R}, M)$ is the unique vector field obtained by applying S to any vector field on $J^1(\mathbb{R}, M)$ which is projectable to X. In particular, $S(X^{(1)}) = X^V$ (see Crampin *et al* 1984). By (ii) one has that $L_{X^{(1)}}\Theta_L = df$ and so $X^{(1)} \lrcorner d\Theta_L = d(f - \langle X^{(1)}, \Theta_L \rangle)$. Hence the first integral associated with $X^{(1)}$, as can be seen by applying Γ to each side of the last equation, is $f - \langle X^{(1)}, \Theta_L \rangle$ which is easily seen to be $f - L\tau - X^V L$, using the definition of Θ_L and the fact that $S(X^{(1)}) = X^V$.

(ii) of proposition 2 was derived by Rund (1972), but by rather different means. The following result is essentially Logan's theorem (Logan 1985).

Corollary. If X is a Noether symmetry of the Euler-Lagrange equation Γ_L, then it is a Lie symmetry of Γ_L.

Proof. Computing the Lie derivative along $X^{(1)}$ of each side of (5) and using the fact that the Lie derivative operator is a derivation one obtains

$$
[X^{(1)}, \Gamma_L] \lrcorner d\Theta_L + \Gamma \lrcorner L_{X^{(1)}}\, d\Theta_L = 0.
$$

But if X is Noether symmetry, then proposition 2(ii) implies that $L_{X^{(1)}}\,\mathrm{d}\Theta_{\mathrm{L}}=0$ and hence $[X^{(1)}, \Gamma_{\mathrm{L}}]$ is characteristic to $\mathrm{d}\Theta_{\mathrm{L}}$ and so must be a multiple of Γ_{L}, i.e. be a Lie symmetry.

The converse to the corollary does not hold, so that not every Lie symmetry is Noether. As to the converse of Logan's theorem, it is not at all clear that any invariant principle underlies it. On the other hand, if X is a Lie symmetry and one wishes to know whether it is Noether, (ii) of proposition 2 provides very simple necessary and sufficient tests. For example, of the two variational problems considered by Logan, the second ($\int t^2(\frac{1}{2}u^2-\frac{1}{6}x^6)\,\mathrm{d}t$—the Euler–Fowler problem) admits $X=t\,\partial/x-\frac{1}{2}x\,\partial/\partial x$ as a Lie symmetry. One may easily check that $X^{(1)}L=-L$ so that X is a Noether symmetry. In the first problem ($\int(xu^3+xt^{-1/2})\,\mathrm{d}t$), $X=\partial/\partial t+\frac{5}{6}x\,\partial/\partial x$ is a Lie symmetry and one easily finds that $L_{X^{(1)}}L\,\mathrm{d}t=\frac{4}{3}L\,\mathrm{d}t$ so that X is not a Noether symmetry. This yields an explanation of the different phenomena occurring in these two problems which surely is at least as convincing as Logan's.

References

Ames W F 1972 *Nonlinear Partial Differential Equations in Engineering* vol II (New York: Academic)
Bluman G W and Cole J D 1974 *Similarity Methods for Differential Equations* (New York: Springer)
Crampin M 1977 *Int. J. Theor. Phys.* **16** 741–54
Crampin M, Prince G E and Thompson G 1984 *J. Phys. A: Math. Gen.* **17** 1437–47
Logan J D 1985 *J. Phys. A: Math. Gen.* **18** 2151–5
Noether E 1918 *Nachr. Gesell. Wissenschaft Göttingen* **2** 235
Prince G E 1983 *Bull. Austr. Math. Soc.* **27** 53–71
Rund H 1972 *Utilitas Math.* **2** 205–14
Sarlet W and Cantrijn F 1982 *SIAM Rev.* **23** 467–94
Sniatycki J 1970 *Proc. Camb. Phil. Soc.* **68** 475–84